Pole figury, całki niewłaściwe.

1. Obliczyć pole obszaru ograniczonego krzywymi (wykonać rysunek): a) $y = x - x^2,\ y = x^2 - 1$ $\left[\frac{9}{8}\right]$

b) $y = x^2 + x - 2,\ y = -x + 1$ $\left[\frac{32}{3}\right]$ c) $y = x^2 - 6x + 10,\ y = 6x - x^2$ $\left[21\frac{1}{3}\right]$ d) $y = \frac{2}{x},\ y = x - 1,\ x = 4$ $\left[4 + \ln\frac{1}{4}\right]$

e) $y = 2 - x,\ y = \sqrt{x},\ x = 0$ $\left[\frac{7}{6}\right]$ f) $y = \frac{x^2}{4},\ y = \frac{8}{x^2 + 4}$ $\left[2\pi - \frac{4}{3}\right]$ g) $y^2 = 2 - x,\ y = -x$ $\left[4\frac{1}{2}\right]$.

2. Obliczyć (lub wykazać rozbieżność) całki niewłaściwe: a) $\int_{-\infty}^{+\infty} \frac{2x\,dx}{x^2 + 1}$ [rozbieżna] b) $\int_1^{\infty} \frac{dx}{x^2 - 6x + 13}$ $\left[\frac{3\pi}{8}\right]$

c) $\int_1^{\infty} \frac{4x + \sqrt{x}}{x^2} dx$ [rozbież.] d) $\int_1^{\infty} \frac{4x + \sqrt{x}}{x^3} dx$ $\left[\frac{14}{3}\right]$ e) $\int_{-1}^{0} \frac{e^{-\frac{1}{x}}}{x^3} dx$ [rozbież.] f) $\int_0^{\frac{1}{e}} \frac{dx}{x \ln^2 x}$ [1] g) $\int_0^{4} \frac{4x}{x^2 \sqrt{x}} dx$ [rozbież.]

h) $\int_0^{1} \frac{\sqrt{x}}{3x} dx$ $\left[\frac{2}{3}\right]$ i) $\int_0^{2} \frac{dx}{(x - 1)^2}$ j) $\int_{-1}^{1} \frac{dx}{\sqrt[3]{x^5}}$ k) $\int_0^{2} \frac{\sqrt{x}}{4x^3} dx$ l) $\int_{-1}^{2} \frac{1}{x^2 - 2x - 3} dx$ [i, j, k, l - całki rozbieżne].

Rozwiązania niektórych zadań.

1. c) $y = x^2 - 6x + 10,\ y = 6x - x^2$; $x^2 - 6x + 10 = 6x - x^2 \Leftrightarrow 2x^2 - 12x + 10 = 0 \Leftrightarrow x = 1$ lub $x = 5$.

$$P = \int_1^5 (6x - x^2 - x^2 + 6x - 10)dx = \int_1^5 (12x - 2x^2 - 10)dx = (6x^2 - \tfrac{2}{3}x^3 - 10x)\Big|_1^5 = 150 - \tfrac{250}{3} - 50 - 6 + \tfrac{2}{3} + 10 =$$
$$= 104 - \tfrac{248}{3} = 21\tfrac{1}{3}$$

d) $y = \frac{2}{x},\ y = x - 1,\ x = 4$; $\frac{2}{x} = x - 1 \Leftrightarrow x^2 - x - 2 = 0 \Leftrightarrow x_1 = -1,\ x_2 = 2$, $P = \int_2^4 (x - 1 - \frac{2}{x})dx = 4 - 2\ln 2$.

$x = -1$ nie jest punktem przecięcia tych krzywch.

f) $y = \frac{x^2}{4},\ y = \frac{8}{x^2 + 4}$ punkty wspólne dla $x = -2$ lub $x = 2$.

$$P = \int_{-2}^{2} \left(\frac{8}{x^2 + 4} - \frac{x^2}{4} \right) dx = 4\mathrm{arctg}\left(\frac{x}{2}\right)\Big|_{-2}^{2} - \frac{x^3}{12}\Big|_{-2}^{2} = 4\mathrm{arctg}1 - 4\mathrm{arctg}(-1) - \frac{2}{3} - \frac{2}{3} = 2\pi - \frac{4}{3}.$$

g) $y^2 = 2 - x,\ y = -x$; punkty wspólne: $\begin{cases} y^2 = 2 - x \\ y = -x \end{cases} \Rightarrow y^2 = 2 + y \Rightarrow y^2 - y - 2 = 0$, stąd $\begin{cases} y = 2 \\ x = -2 \end{cases}$ lub

$\begin{cases} y = -1 \\ y = 1 \end{cases}$. $P = \int_{-2}^{1} (\sqrt{2 - x} - (-x))dx + \int_1^2 (\sqrt{2 - x} - (-\sqrt{2 - x}))dx = \int_{-2}^{1} (2 - x)^{\frac{1}{2}} dx + \int_{-2}^{1} x\,dx + 2\int_1^2 (2 - x)^{\frac{1}{2}} dx =$

$$= \frac{2}{3}(-1)(2 - x)^{\frac{3}{2}}\Big|_{-2}^{1} + \frac{x^2}{2}\Big|_{-2}^{1} - \frac{4}{3}(2 - x)^{\frac{3}{2}}\Big|_1^2 = \frac{2}{3}\cdot 7 + \frac{1}{2} - \frac{4}{2} + \frac{4}{3} = \frac{14}{3} - \frac{3}{2} + \frac{4}{3} = \frac{9}{2}$$

2. a) $\int_{-\infty}^{+\infty} \frac{2x\,dx}{x^2 + 1} = \int_{-\infty}^{0} \frac{2x\,dx}{x^2 + 1} + \int_0^{+\infty} \frac{2x\,dx}{x^2 + 1} = \lim_{a \to -\infty} \int_a^0 \frac{2x\,dx}{x^2 + 1} + \lim_{b \to +\infty} \int_0^b \frac{2x\,dx}{x^2 + 1} = \lim_{a \to -\infty} \ln(x^2 + 1) + \lim_{b \to +\infty} \ln(x^2 + 1) = \infty + \infty$,

więc całka rozbieżna.

b) $\int_1^\infty \frac{dx}{x^2-6x+13} = \lim_{a\to\infty}\left(\frac{1}{2}\operatorname{arctg}\left(\frac{x-3}{2}\right)\right)\Big|_1^a = \lim_{a\to\infty}\left(\frac{1}{2}\operatorname{arctg}\left(\frac{a-3}{2}\right)-\frac{1}{2}\operatorname{arctg}(-1)\right) = \frac{1}{2}\cdot\frac{\pi}{2}-\frac{1}{2}\cdot\frac{\pi}{4}=\frac{3\pi}{8}.$

c) $\int_1^\infty \frac{4x+\sqrt{x}}{x^2}dx = \lim_{a\to\infty}\int_1^a \frac{4x+\sqrt{x}}{x^2}dx = \lim_{a\to\infty}(4\ln x - \frac{2}{\sqrt{x}})\Big|_1^a = \lim_{a\to\infty}(4\ln a - \frac{2}{\sqrt{a}} - 4\ln 1 + 2) = \infty$, więc całka rozbieżna.

d) $\int_1^\infty \frac{4x+\sqrt{x}}{x^3}dx = \lim_{a\to\infty}\int_1^a \frac{4x+\sqrt{x}}{x^3}dx = \lim_{a\to\infty}\left(-\frac{4}{x}-\frac{2}{3\sqrt{x^3}}\right)\Big|_1^a = \lim_{a\to\infty}\left(-\frac{4}{a}-\frac{2}{3\sqrt{a^3}}+4+\frac{2}{3}\right) = \frac{14}{3}.$

f) $\int_0^{\frac{1}{e}} \frac{dx}{x\ln^2 x} = \lim_{a\to 0^+}\left(-\frac{1}{\ln x}\right)\Big|_a^{\frac{1}{e}} = \lim_{a\to 0^+}\left(-\frac{1}{\ln e^{-1}}+\frac{1}{\ln a}\right) = -(-1)+0=1$

$$\int\frac{dx}{x\ln^2 x} = \begin{bmatrix} \ln x = t \\ \frac{1}{x}dx = dt \end{bmatrix} = \int\frac{dt}{t^2} = -\frac{1}{t} = -\frac{1}{\ln x}.$$

i) $\int_0^2 \frac{dx}{(x-1)^2} = \int_0^1 \frac{dx}{(x-1)^2} + \int_1^2 \frac{dx}{(x-1)^2} = \lim_{a\to 1^-}\int_0^a \frac{dx}{(x-1)^2} + \lim_{a\to 1^+}\int_a^1 \frac{dx}{(x-1)^2} = \lim_{a\to 1^-}\left(-\frac{1}{x-1}\Big|_0^a\right) + \lim_{a\to 1^+}\left(-\frac{1}{x-1}\Big|_a^0\right) =$

$\lim_{a\to 1^-}\left(-\frac{1}{a-1}-1\right) + \lim_{a\to 1^+}\left(1+\frac{1}{a-1}\right) = \infty + \infty = \infty$ - całka rozbieżna.

l) $\int_{-1}^2 \frac{1}{x^2-2x-3}dx = \lim_{a\to -1^+}\left(-\frac{1}{4}\ln|x+1|+\frac{1}{4}\ln|x-3|\right)\Big|_a^2 = \lim_{a\to -1^+}\left(-\frac{1}{4}\ln 3+\frac{1}{4}\ln|2-3|+\frac{1}{4}\ln(a+1)-\frac{1}{4}\ln|a-3|\right) =$

$= -\frac{1}{4}\ln 12 - \infty = -\infty$ całka rozbieżna.

$$\frac{1}{x^2-2x-3} = \frac{1}{(x+1)(x-3)} = \frac{-\frac{1}{4}}{x+1}+\frac{\frac{1}{4}}{x-3}\ ;\ \int\frac{1}{x^2-2x-3}dx = -\frac{1}{4}\ln|x+1|+\frac{1}{4}\ln|x-3|.$$