CAŁKI NIEOZNACZONE.

1. Obliczyć całki: a) $\int \frac{(x+\sqrt{x})^2}{x^3}dx$ b) $\int \frac{x^2+2x^2\sqrt{x}+2}{x^3}dx$ c) $\int \frac{\sqrt{x}-2}{x\sqrt{x}}dx$ d) $\int \frac{2x+1}{x^2+1}dx$ e) $\int \frac{3x-1}{x^2+1}dx$.

2. Całkując przez części, obliczyć: a) $\int \operatorname{arctg} x\,dx$ b) $\int x^5 \ln x\,dx$ c) $\int \frac{\ln x}{x^3}dx$ d) $\int \ln x\,dx$ e) $\int \ln^2 x\,dx$

f) $\int x^2 \operatorname{arctg} x dx$ g) $\int x\ln(x^2+1)\,dx$.

3. Używając podstawień, obliczyć całki: a) $\int \frac{dx}{e^x+e^{-x}}$ b) $\int x\cdot\sqrt{4-x^2}dx$ c) $\int \frac{\cos x}{1+4\sin^2 x}dx$ d) $\int e^{\sqrt{x}}dx$

e) $\int \frac{e^{\frac{1}{x}}}{x^3}dx$ f) $\int \frac{\operatorname{ctg} x}{\ln(\sin x)}dx$ g) $\int \frac{\cos x}{\sqrt{5+3\sin x}}dx$ h) $\int \frac{x\,dx}{\sqrt{1-x^4}}$ i) $\int \frac{\operatorname{tg} x}{1+\operatorname{tg}^4 x}\cdot\frac{dx}{\cos^2 x}$ j) $\int \frac{\ln \operatorname{tg} x}{\sin x\cdot\cos x}dx$.

d) $\int \frac{x+3}{\sqrt{2+x}}dx$ e) $\int \frac{\sin^3 x}{\cos^2 x}dx$ f) $\int \frac{\cos x}{\sqrt{1+\sin x}}dx$ g) $\int \frac{\cos x}{\sin^4 x}dx$ j) $\int \frac{dx}{x\sqrt{2+\ln x}}$ k) $\int \frac{\sqrt[3]{\operatorname{tg} x+2}}{\cos^2 x}dx$

l) $\int \frac{\sin^3 x}{\cos^2 x}dx$ m) $\int \sin^6 x\cos^5 x\,dx$ n) $\int \sin^5 x\,dx$ o) $\int \operatorname{tg}^3 x\,dx$ p) $\int \frac{dx}{\sin x\cos x}$.

4. Znaleźć całki funkcji wymiernych: a) $\int \frac{dx}{4x^2+9}$ b) $\int \frac{x+5}{x^2+4x+8}dx$ c) $\int \frac{dx}{x^2-8x+20}$

d) $\int \frac{3x+1}{x^2+6x+13}dx$ e) $\int \frac{x\,dx}{x^2+2x-3}$ f) $\int \frac{x^2+7x}{x^2+5x-6}dx$ g) $\int \frac{x^2+6}{x^2+3x-4}dx$ h) $\int \frac{x^2}{x^4-1}dx$

i) $\int \frac{dx}{x^3-7x-6}$ j) $\int \frac{dx}{x^3+x}$ k) $\int \frac{4x+3}{x^3-x^2-6x}dx$.

ROZWIĄZANIA NIEKTÓRYCH ZADAŃ.

1. a) $\int \frac{(x+\sqrt{x})^2}{x^3}dx = \int \frac{x^2+2x\sqrt{x}+x}{x^3} = \int \frac{1}{x}dx + 2\int x^{-\frac{3}{2}}dx + \int x^{-2}dx = \ln|x| - 4x^{-\frac{1}{2}} - x^{-1} = \ln|x| - \frac{4}{\sqrt{x}} - \frac{1}{x^2}$.

d) $\int \frac{2x+1}{x^2+1}dx = \int \frac{2x}{x^2+1}dx + \int \frac{1}{x^2+1}dx = \ln(x^2+1) + \operatorname{arctg} x$.

2. a) $\int \operatorname{arctg} x\,dx = \left| \begin{matrix} u=\operatorname{arctg} x & u'=\frac{1}{x^2+1} \\ v'=1 & v=x \end{matrix} \right| = x\operatorname{arctg} x - \int \frac{x}{x^2+1}dx = x\operatorname{arctg} x - \tfrac{1}{2}\ln(x^2+1)$.

c) $\int \frac{\ln x}{x^3}dx == \left| \begin{matrix} u=\ln x & u'=\frac{1}{x} \\ v'=\frac{1}{x^3} & v=-\frac{1}{2x^2} \end{matrix} \right| = -\frac{1}{2x^2}\ln x + \tfrac{1}{2}\int \frac{1}{x^3}dx = -\frac{1}{2x^2}\ln x + \frac{1}{2}(-\frac{1}{2x^2}) = -\frac{1}{2x^2}\ln x - \frac{1}{4x^2}$.

e) $\int \ln^2 x\,dx = \left| \begin{matrix} u=\ln^2 x & u'=\frac{1}{x}\cdot 2\ln x \\ v'=1 & v=x \end{matrix} \right| = x\ln^2 x - 2\int \ln x\,dx = \left| \begin{matrix} u=\ln x & u'=\frac{1}{x} \\ v'=1 & v=x \end{matrix} \right| = x\ln^2 x - 2(x\ln x - \int dx)$

$= x\ln^2 x - 2x\ln x + 2x$.

2. g) $\int x\ln(x^2+1)\,dx = \left| \begin{matrix} u=\ln(x^2+1) & u'=\frac{2x}{x^2+1} \\ v'=x & v=\frac{x^2}{2} \end{matrix} \right| = \frac{x^2}{2}\ln(x^2+1) - \int \frac{x^3}{x^2+1}dx =$

$$=\frac{x^2}{2}\ln(x^2+1)-\int\frac{x^3+x-x}{x^2+1}dx=\frac{x^2}{2}\ln(x^2+1)-\int\frac{x^3+x}{x^2+1}dx+\int\frac{x}{x^2+1}dx=\frac{x^2}{2}\ln(x^2+1)-\frac{x^2}{2}+\frac{1}{2}\ln(x^2+1).$$

3. a) $\int\frac{dx}{e^x+e^{-x}}=\int\frac{e^x dx}{e^{2x}+1}\left[\begin{matrix} t=e^x \\ dt=e^x dx\end{matrix}\right]=\int\frac{dt}{t^2+1}=\operatorname{arctg} t=\operatorname{arctg} e^x$.

d) $\int e^{\sqrt{x}}dx=\left[\begin{matrix} t=\sqrt{x} \\ t^2=x \\ 2tdt=dx\end{matrix}\right]=\int e^t 2tdt=\left[\begin{matrix} u=2t & u'=2 \\ v'=e^t & v=e^t\end{matrix}\right]=2te^t-\int 2e^t dt=2te^t-2e^t=2\sqrt{x}\cdot e^{\sqrt{x}}-2e^{\sqrt{x}}$.

e) $\int\frac{e^{\frac{1}{x}}}{x^3}dx=\left[\begin{matrix} \frac{1}{x}=t \\ \frac{1}{x^2}dx=-dt\end{matrix}\right]=\int t\cdot e^t dt=$ dalej przez części jak w d).

f) $\int\frac{\operatorname{ctg} x}{\ln(\sin x)}dx=\left[\begin{matrix} t=\ln(\sin x) \\ dt=\frac{1}{\sin x}\cdot\cos x dx=\operatorname{ctg} x\, dx\end{matrix}\right]=\int\frac{dt}{t}=\ln t=\ln(\ln(\sin x))$.

h) $\int\frac{x\,dx}{\sqrt{1-x^4}}=\left[\begin{matrix} x^2=t \\ xdx=\frac{1}{2}dt\end{matrix}\right]=\frac{1}{2}\arcsin t=\frac{1}{2}\arcsin x^2$

k) $\int\frac{\sqrt[3]{\operatorname{tg} x+2}}{\cos^2 x}dx=\left|\begin{matrix} t=\sqrt[3]{\operatorname{tg} x+2} \\ t^3=\operatorname{tg} x+2 \\ 3t^2dt=\frac{dx}{\cos^2 x}\end{matrix}\right|=\int t\cdot 3t^2dt=3\int t^3dt=\frac{3}{4}t^4=\frac{3}{4}\sqrt[3]{(\operatorname{tg} x+2)^4}$.

n) $\int\sin^5 x\,dx=\int(\sin^2 x)^2\sin x\,dx=\int(1-\cos^2 x)^2\sin x dx=\left[\begin{matrix} \cos x=t \\ \sin x dx=-dt\end{matrix}\right]=-\int(1-t^2)^2dt=$

$=-\int dt+2\int t^2dt-\int t^4dt=-t+\frac{2}{3}t^3-\frac{1}{5}t^5=-\sin x+\frac{2}{3}\sin^3 x-\frac{1}{5}\sin^5 x$.

o) $\int\operatorname{tg}^3 x\,dx=\int\frac{\sin^3 x}{\cos^3 x}dx=\int\frac{1-\cos^2 x}{\cos^3 x}\sin x\,dx=\left[\begin{matrix} \cos x=t \\ \sin x dx=-dt\end{matrix}\right]=-\int\frac{1-t^2}{t^3}dt=-\int t^{-3}dt+\int\frac{dt}{t}=\frac{t^{-2}}{2}+\ln|t|=$

$=\frac{1}{\cos^2 x}+\ln|\cos x|$

p) $\int\frac{dx}{\sin x\cos x}=\int\frac{\cos x dx}{\sin x\cos^2 x}=\int\operatorname{ctg} x\cdot\frac{dx}{\cos^2 x}=\int\frac{1}{\operatorname{tg} x}\cdot\frac{dx}{\cos^2 x}=\left|\begin{matrix} t=\operatorname{tg} x \\ dt=\frac{dx}{\cos^2 x}\end{matrix}\right|=\int\frac{1}{t}dt=\ln t=\ln(\operatorname{tg}x)$.

4. W przykładach a) - d) jest $\Delta<0$. a) $\int\frac{dx}{4x^2+9}=\frac{1}{9}\int\frac{dx}{(\frac{2}{3}x)^2+1}=\frac{1}{6}\operatorname{arctg}(\frac{2}{3}x)$.

b) $\int\frac{x+5}{x^2+4x+8}dx=\int\frac{x+2}{x^2+4x+8}dx+\int\frac{3}{x^2+4x+8}dx=\frac{1}{2}\int\frac{2x+4}{x^2+4x+8}dx+\int\frac{3}{(\frac{x+2}{2})^2+1}dx=$

$=\ln(x^2+4x+8)+\frac{1}{2}\operatorname{arctg}(\frac{x+2}{2})$.

d) $\int\frac{3x+1}{x^2+6x+13}dx=\int\frac{3x}{x^2+6x+13}dx+\int\frac{1}{x^2+6x+13}=\frac{3}{2}\int\frac{2x+6-6}{x^2+6x+13}dx+\int\frac{1}{x^2+6x+13}dx=$

$\frac{3}{2}\int\frac{2x+6}{x^2+6x+13}dx-8\int\frac{1}{(x+3)^2+4}dx=\ln(x^2+6x+13)-4\operatorname{arctg}(\frac{x+3}{2})$.

e) $\int\frac{x\,dx}{x^2+2x-3}:\Delta>0,\ x^3+2x-3=(x-1)(x+3),\int\frac{x\,dx}{x^2+2x-3}=\frac{1}{4}\int\frac{dx}{x-1}+\frac{3}{4}\int\frac{dx}{x+3}=\frac{1}{4}\ln|x-1|+\frac{3}{4}\ln|x+3|$.

g) $\int\frac{x^2+6}{x^2+3x-4}dx=\int\frac{x^2+3x-4-3x+10}{x^2+3x-4}dx=\int\frac{x^2+3x-4}{x^2+3x-4}dx+\int\frac{-3x+10}{x^2+3x-4}dx$

$=\int dx+\int\frac{\frac{7}{5}}{x-1}dx-\int\frac{\frac{22}{5}}{x+4}dx=x+\frac{7}{5}\ln|x-1|-\frac{22}{5}\ln|x+4|$.

h) $\int \frac{x^2}{x^4-1}dx = \frac{1}{4}\int \frac{dx}{x-1} - \frac{1}{4}\int \frac{dx}{x+1} + \frac{1}{2}\int \frac{dx}{x^2+1} = \frac{1}{4}\ln(x-1) + \frac{1}{2}\ln(x+1) + \frac{1}{2}\operatorname{arctg} x$.

i) $\int \frac{dx}{x^3-7x-6}$; $x^3-7x-6=(x+1)(x+2)(x-3)$;

$$\int \frac{dx}{x^3-7x-6} = -\frac{1}{4}\int \frac{dx}{x+1} + \frac{1}{5}\int \frac{dx}{x+2} + \frac{1}{20}\int \frac{dx}{x-3} = -\frac{1}{4}\ln\left|x+1\right| + \frac{1}{5}\ln\left|x+2\right| + \frac{1}{20}\ln(x-3) .$$