CAŁKI OZNACZONE, POLE FIGURY.

1. Znaleźć całki (w [.] są wyniki): a) $\int_2^4 \frac{x+2}{x^2-4x+8}$ $\left[\ln\sqrt{2}+\frac{\pi}{2}\right]$ b) $\int_0^1 x\ln(x^2+1)\,dx$ $\left[\ln 2-\frac{1}{2}\right]$ (p.części)

c) $\int_0^{\frac{\pi}{4}} \mathrm{tg}^2 x\,dx$ $\left[1-\frac{\pi}{4}\right]$ d) $\int_0^{\frac{\pi}{2}} \sin^3 x\,dx$ $\left[\frac{2}{3}\right]$ e) $\int_0^{\frac{\pi}{2}} \sin^5 x\cdot\cos^3 x\,dx$ $\left[\frac{1}{24}\right]$ f) $\int_0^{\frac{\pi}{3}} \mathrm{tg}^3 x\,dx$ $\left[\frac{2}{3}-\ln 2\right]$

g) $\int_0^{\sqrt{\frac{\pi}{2}}} x\sin x^2 dx$ $\left[\frac{1}{2}\right]$ h) $\int_0^{\frac{\pi}{3}} \cos^2 x\,dx$ $\left[\frac{\sqrt{3}}{8}+\frac{\pi}{6}\right]$ i) $\int_0^{\sqrt{3}} \frac{x\,dx}{\sqrt{x^2+1}}$ [1] j) $\int_0^5 \frac{x+2}{\sqrt{x+4}}dx$ $\left[\frac{26}{3}\right]$

k) $\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{2+\sin x}}dx$ $\left[2\sqrt{3}-2\sqrt{2}\right]$ l) $\int_{\sqrt{2}}^{\sqrt{5}} \frac{x}{\sqrt{x^2-1}}dx$ $[1]$ m) $\int_{-\frac{1}{2}}^{1} \frac{dx}{\sqrt{8+2x-x^2}}$ $\left[\frac{\pi}{6}\right]$

l) $\int_1^{\sqrt{3}} x\,\mathrm{arc\,tg}\frac{1}{x}dx$ $\left[\frac{\pi}{12}+\frac{1}{2}\cdot(\sqrt{3}-1)\right]$.

2. Obliczyć pole obszaru ograniczonego krzywymi (wykonać rysunek): a) $y=x-x^2,\ y=x^2-1$ $\left[\frac{9}{8}\right]$

b) $y=x^2+x-2,\ y=-x+1$ $\left[\frac{32}{3}\right]$ c) $y=x^2-6x+10$, $y=6x-x^2$ $\left[21\frac{1}{3}\right]$ d) $y=\frac{2}{x},\ y=x-1,\ x=4$ $\left[4+\ln\frac{1}{4}\right]$

e) $y=2-x,\ y=\sqrt{x},\ x=0$ $\left[\frac{7}{6}\right]$ f) $y=\frac{x^2}{4}$, $y=\frac{8}{x^2+4}$ $\left[2\pi-\frac{4}{3}\right]$ g) $y^2=2-x,\ y=-x$ $\left[4\frac{1}{2}\right]$.

Rozwiązania niektórych zadań.

1. a) $$\int_2^4 \frac{x+2}{x^2-4x+8}dx=\frac{1}{2}\int_2^4\frac{2x-4+8}{x^2-4x+8}dx=\frac{1}{2}\int_2^4\frac{2x-4}{x^2-4x+8}dx+\ 4\int_2^4\frac{dx}{(x-2)^2+4}=$$
$$\frac{1}{2}\ln(x^2-4x+8)\Big|_2^4+2\,\mathrm{arc\,tg}\left(\frac{x-2}{2}\right)\Big|_2^4=\frac{1}{2}(\ln 8-\ln 4)+2\,\mathrm{arc\,tg}1\ =\ \ln\sqrt{2}+\frac{\pi}{2}$$

b) $$\int_0^1 x\ln(x^2+1)\,dx=\begin{bmatrix} u=\ln(x^2+1) & u'=\dfrac{2x}{x^2+1} \\ v'=x & v=\dfrac{x^2}{2}\end{bmatrix}=\frac{x^2}{2}\ln(x^2+1)\Big|_0^1-\int_0^1\frac{x^3}{x^2+1}dx=$$
$$=\frac{1}{2}\ln 2-\left[\int_0^1 x\,dx-\int_0^1\frac{x}{x^2+1}dx\right]=\frac{1}{2}\ln 2-\frac{x^2}{2}\Big|_0^1+\frac{1}{2}\ln(x^2+1)\Big|_0^1=\ln 2-0{,}5\,.$$

c) $$\int \mathrm{tg}^2 x\,dx=\int\frac{1-\cos^2 x}{\cos^2 x}dx=\int\frac{1}{\cos^2 x}dx-\int dx=\mathrm{tg}\,x-x\,;\ \int_0^{\frac{\pi}{4}}\mathrm{tg}^2 x\,dx=\mathrm{tg}\,x-x\Big|_0^{\frac{\pi}{4}}=1-\frac{\pi}{4}\ .$$

f) $$\int_0^{\frac{\pi}{3}}\mathrm{tg}^3 x\,dx=\begin{bmatrix} t=\mathrm{tg}\,x \\ \mathrm{arctg}\,t=x \\ \dfrac{dt}{1+t^2}=dx\end{bmatrix}=\int_0^{\sqrt{3}}\frac{t^3}{1+t^2}dt=\int_0^{\sqrt{3}}\frac{t^3+t-t}{1+t^2}dt=\int_0^{\sqrt{3}}\frac{t^3+t}{1+t^2}dt-\tfrac{1}{2}\int_0^{\sqrt{3}}\frac{2t}{t^2+1}\ =\int_0^{\sqrt{3}}t\,dt-\tfrac{1}{2}\ln(t^2+1)\ \Big|_0^{\sqrt{3}}=$$

x	0	$\frac{\pi}{3}$
t	0	$\sqrt{3}$

$$= \left.\frac{t^2}{2}\right|_0^{\sqrt{3}} - \tfrac{1}{2}\ln 4 - \tfrac{1}{2}\ln 1 = \frac{3}{2} - \ln 2 .$$

g) $$\int_0^{\sqrt{\frac{\pi}{2}}} x\sin x^2 dx = \begin{bmatrix} x^2 = t \\ 2xdx = dt \end{bmatrix} = \tfrac{1}{2}\int_0^{\frac{\pi}{2}} \sin t\, dt = -\tfrac{1}{2}\cos t\Big|_0^{\frac{\pi}{2}} = 0 + \tfrac{1}{2}\cdot 1 = \frac{1}{2} .$$

x	0	$\sqrt{\frac{\pi}{2}}$
t	0	$\frac{\pi}{2}$

h) $$\int_0^{\frac{\pi}{3}} \cos^2 x\, dx = \left[\cos^2 x = \frac{1+\cos 2x}{2}\right] = \tfrac{1}{2}\int_0^{\frac{\pi}{3}} (1+\cos 2x)dx = \tfrac{1}{2}x\Big|_0^{\frac{\pi}{3}} + \tfrac{1}{4}\sin 2x\Big|_0^{\frac{\pi}{3}} = \frac{\pi}{6} + \frac{1}{4}\sin\frac{2\pi}{3} = \frac{\pi}{6} + \frac{1}{4}\cdot\frac{\sqrt{3}}{2} =$$

$$= \frac{\pi}{6} + \frac{\sqrt{3}}{8} .$$

i) $$\int_0^{\sqrt{3}} \frac{xdx}{\sqrt{x^2+1}} = \begin{bmatrix} \sqrt{x^2+1} = t \\ x^2+1 = t^2 \\ xdx = tdt \end{bmatrix} = \int_1^2 \frac{t\,dt}{t} = t\Big|_1^2 = 2-1 = 1$$

x	0	$\sqrt{3}$
t	1	2

k) $$\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{2+\sin x}} dx = \begin{bmatrix} t = \sqrt{2+\sin x} \\ t^2 = 2+\sin x \\ 2tdt = \cos x dx \end{bmatrix} = \int_{\sqrt{2}}^{\sqrt{3}} \frac{2tdt}{t} = 2t\Big|_{\sqrt{2}}^{\sqrt{3}} = 2\sqrt{3} - 2\sqrt{2} .$$

x	0	$\frac{\pi}{2}$
t	$\sqrt{2}$	$\sqrt{3}$

m) $$\int_{-\frac{1}{2}}^{1} \frac{dx}{\sqrt{8+2x-x^2}} = \int_{-\frac{1}{2}}^{1} \frac{dx}{\sqrt{9-(x-1)^2}} = \frac{1}{3}\int_{-\frac{1}{2}}^{1} \frac{dx}{\sqrt{1-(\frac{x-1}{3})^2}} = \arcsin(\tfrac{x-1}{3})\Big|_{-\frac{1}{2}}^{1} = \arcsin 0 - \arcsin(-\tfrac{1}{2}) = 0 - \left(-\frac{\pi}{6}\right) = \frac{\pi}{6} .$$

2\. c) $y = x^2 - 6x + 10$, $y = 6x - x^2$; $x^2 - 6x + 10 = 6x - x^2 \Leftrightarrow 2x^2 - 12x + 10 = 0 \Leftrightarrow x = 1$ lub $x = 5$.

$$P = \int_1^5 (6x - x^2 - x^2 + 6x - 10)dx = \int_1^5 (12x - 2x^2 - 10)dx = (6x^2 - \tfrac{2}{3}x^3 - 10x)\Big|_1^5 = 150 - \tfrac{250}{3} - 50 - 6 + \tfrac{2}{3} + 10 =$$

$$= 104 - \tfrac{248}{3} = 21\tfrac{1}{3}$$

d) $y = \frac{2}{x}$, $y = x - 1$, $x = 4$; $\frac{2}{x} = x - 1 \Leftrightarrow x^2 - x - 2 = 0 \Leftrightarrow x_1 = -1, x_2 = 2$, $P = \int_2^4 (x - 1 - \frac{2}{x})dx = 4 - 2\ln 2$.

$x = -1$ nie jest punktem przecięcia tych krzywch.

f) $y = \frac{x^2}{4}$, $y = \frac{8}{x^2+4}$ punkty wspólne dla $x = -2$ lub $x = 2$.

$$P = \int_{-2}^{2}\left(\frac{8}{x^2+4} - \frac{x^2}{4}\right)dx = 4\mathrm{arctg}\left(\frac{x}{2}\right)\Big|_{-2}^{2} - \frac{x^3}{12}\Big|_{-2}^{2} = 4\mathrm{arctg}1 - 4\mathrm{arctg}(-1) - \frac{2}{3} - \frac{2}{3} = 2\pi - \frac{4}{3} .$$

g) $y^2 = 2 - x$, $y = -x$; punkty wspólne: $\begin{cases} y^2 = 2 - x \\ y = -x \end{cases} \Rightarrow y^2 = 2 + y \Rightarrow y^2 - y - 2 = 0$, stąd $\begin{cases} y = 2 \\ x = -2 \end{cases}$ lub

$\begin{cases} y = -1 \\ y = 1 \end{cases}$. $P = \int_{-2}^{1} (\sqrt{2-x} - (-x))dx + \int_{1}^{2} (\sqrt{2-x} - (-\sqrt{2-x}\,))dx = \int_{-2}^{1} (2-x)^{\frac{1}{2}} dx + \int_{-2}^{1} x\,dx + 2\int_{1}^{2} (2-x)^{\frac{1}{2}} dx =$

$= \frac{2}{3}(-1)(2-x)^{\frac{3}{2}} \Big|_{-2}^{1} + \frac{x^2}{2} \Big|_{-2}^{1} - \frac{4}{3}(2-x)^{\frac{3}{2}} \Big|_{1}^{2} = \frac{2}{3} \cdot 7 + \frac{1}{2} - \frac{4}{2} + \frac{4}{3} = \frac{14}{3} - \frac{3}{2} + \frac{4}{3} = \frac{9}{2}$